9 Graph Algorithms

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Learning Outcomes

Unit 9: ***Graph Algorithms***

1. Know basic terminology from graph theory, including types of graphs.
2. Know adjacency matrix and adjacency list representations and their performance characteristica.
3. Know graph-traversal based algorithm, including efficient implementations.
4. Be able to proof correctness of graph-traversal-based algorithms.
5. Know algorithms for maximum flows in networks.
6. Be able to model new algorithmic problems as graph problems.

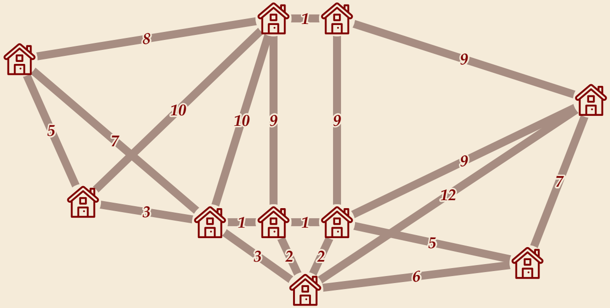
Outline

9 Graph Algorithms

9.1 Introduction & Definitions

Graphs in real life

- a graph is an abstraction of *entities* with their (pairwise) *relationships*
- abundant examples in real life (often called network there)
  - social networks: e. g. persons and their friendships, . . .   *Five/Six? degrees of separation*
  - physical networks: cities and highways, roads networks, power grids etc., the Internet, . . .
  - content networks: world wide web, ontologies, . . .
  - . . .



Many More examples, e. g., in Sedgewick & Wayne's videos:

`https://www.coursera.org/learn/algorithms-part2`

Flavors of Graphs

- Since graphs are used to model so many different entities and relations, they come in several variants

Property	Yes	No
edges are one-way	*directed* graph (*digraph*)	*undirected* graph
≤ 1 edge between u and v	*simple* graph	*multigraph* / with *parallel* edges
edges can lead from v to v	with *loops*	(loop-free)
edges have weights	*(edge-) weighted* graph	*unweighted* graph

🙃 any combination of the above can make sense . . .

- Synonyms:
  - **vertex** („Knoten“) = node = point = „Ecke”
  - **edge** („Kante“) = arc = line = relation = arrow = „Pfeil“
  - **graph** = network

Graph Theory

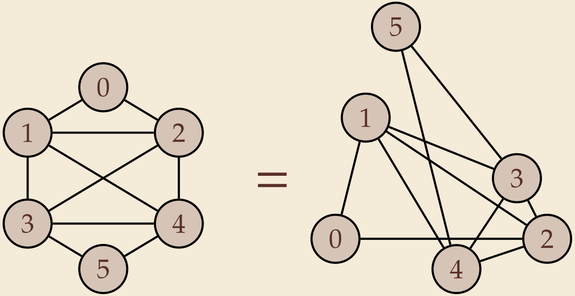
- default: unweighted, undirected, loop-free & simple graphs
- *Graph* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq [V]^2$ a set of *edges*, which are 2-subsets of V: $[V]^2 = \{e : e \subseteq V \wedge |e| = 2\}$

Example

$$V = \{0, 1, 2, 3, 4, 5\}$$

$$E = \{\{0,1\}, \{1,2\}, \{1,4\}, \{1,3\}, \{0,2\}, \{2,4\}, \{2,3\}, \{3,4\}, \{3,5\}, \{4,5\}\}.$$

Graphical representation



like so . . .

. . . or so

(same graph)

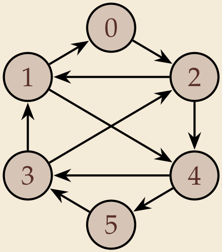
Digraphs

- default digraph: unweighted, loop-free & simple
- *Digraph (directed graph)* $G = (V, E)$ with
  - V a finite of *vertices*
  - $E \subseteq V^2 \setminus \{(v, v) : v \in V\}$ a set of *(directed) edges*,
    $V^2 = V \times V = \{(x, y) : x \in V \wedge y \in V\}$ 2-tuples / ordered pairs over V

Example

$$
\begin{aligned}
V &= \{0, 1, 2, 3, 4, 5\} \\
E &= \{(0,2), (1,0), (1,4), (2,1), (2,4), \\
&\quad (3,1), (3,2), (4,3), (4,5), (5,3)\}
\end{aligned}
$$

Graphical representation



Graph Terminology

Undirected Graphs

- $V(G)$ set of vertices, $E(G)$ set of edges
- write uv (or vu) for edge $\{u, v\}$
- edges *incident* at vertex v: $E(v)$
- u and v are *adjacent* iff $\{u, v\} \in E$,
- *neighborhood* $N(v) = \{w \in V : w \text{ adjacent to } v\}$
- *degree* $d(v) = |E(v)|$

Directed Graphs (where different)

- uv for (u, v)
- iff $(u, v) \in E \vee (v, u) \in E$
- in-/out-neighbors $N_{\text{in}}(v)$, $N_{\text{out}}(v)$
- in-/out-degree $d_{\text{in}}(v)$, $d_{\text{out}}(v)$

- *walk* („*Weg*“) $w[0..n]$ of length n: sequence of vertices with $\forall i \in [0..n) : w[i]w[i+1] \in E$
- *path* („*Pfad*“) p is a (vertex-) simple walk: no duplicate vertices except possibly its endpoints
- *edge-simple* walk: no edge used twice
- *cycle* c is a closed path, i. e., $c[0] = c[n]$

- G is *connected* iff for all $u \neq v \in V$ there is a path from u to v
- G is *acyclic* iff $\nexists$ cycle (of length $n \geq 1$) in G

- *strongly connected* for digraphs (*weakly connected* = connected ignoring directions)

Typical graph-processing problems

- **Path**: Is there a path between s and t?
  Shortest path: What is the shortest path (distance) between s and t?
- **Cycle**: Is there a cycle in the graph?
  Euler tour: Is there a cycle that uses each edge exactly once?
  Hamilton(ian) cycle: Is there a cycle that uses each vertex exactly once.
- **Connectivity**: Is there a way to connect all of the vertices?
  MST: What is the best way to connect all of the vertices?
  Biconnectivity: Is there a vertex whose removal disconnects the graph?
- **Planarity**: Can you draw the graph in the plane with no crossing edges?
- **Graph isomorphism**: Are two graphs the same up to renaming vertices?

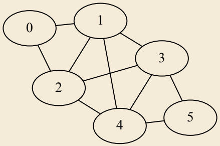
can vary a lot, despite superficial similarity of problems

Challenge: Which of these problems
can be computed in (near) linear time?
in reasonable polynomial time?
are intractable?

Tools to work with graphs

- Convenient GUI to edit & draw graphs: *yEd live*
  `yworks.com/yed-live`
- *graphviz* cmdline utility to draw graphs
  - Simple text format for graphs: `DOT`

```
graph G {
     0 -- 2;     2 -- 4;
     1 -- 0;     2 -- 3;
     1 -- 4;     3 -- 4;
     1 -- 3;     3 -- 5;
     2 -- 1;     4 -- 5;
}
```




```
dot -Tpdf graph.dot -Kfdp > graph.pdf
```

- graphs are typically not built into programming languages, but libraries exist
  - e. g. part of *Google Guava* for Java
  - they usually allow arbitrary objects as vertices
  - aimed at ease of use

9.2 Graph Representations

Graphs in Computer Memory

- We defined graphs in set-theoretic terms...
  but computers can't directly deal with sets efficiently

$\rightsquigarrow$ need to choose a *representation* for graphs.

- which is better depends on the required operations

Key Operations:

- `isAdjacent(`u`,`v`)`
  Test whether $uv \in E$
- `adj(`v`)`
  Adjacency list of v (iterate through (out-) neighbors of v)
- most others can be computed based on these

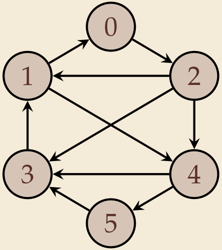
Conventions:

- (di)graph $G = (V, E)$ (omitted if clear from context)
- $n = |V|, \; m = |E|$
- in implementations assume $V = [0..n)$ (if needed, use symbol table to map complex objects to V)

Adjacency Matrix Representation

- adjacency matrix $A \in \{0,1\}^{n\times n}$ of G: matrix with $A[u,v] = [uv \in E]$
  - works for both directed and undirected graphs (undirected $\rightsquigarrow A = A^T$ symmetric)
  - can use a weight $w(uv)$ or multiplicity in $A[u,v]$ instead of $0/1$
  - can represent loops via $A[v,v]$

Example:



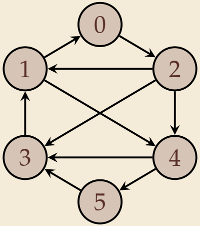
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

👍 `isAdjacent` in $O(1)$ time

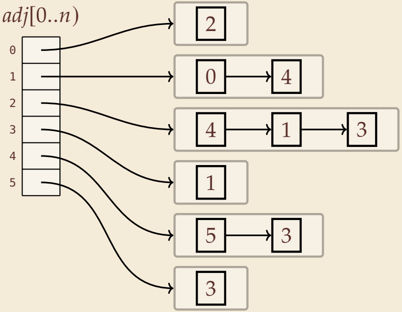
👎 $O(n^2)$ (bits of) space wasteful for sparse graphs

👎 `adj(`v`)` iteration takes $O(n)$ (independent of $d(v)$)

Adjacency List Representation



- Store a linked list of neighbors for each vertex v:
  - $adj[0..n)$ bag of neighbors (as linked list)
  - undirected edge $\{u, v\}$ $\rightsquigarrow$ v in $adj[u]$ and u in $adj[v]$
  - weighted edge uv $\rightsquigarrow$ store pair $(v, w(uv))$ in $adj[u]$
  - multiple edges and loops can be represented



`isAdjacent(`u`,`v`)` takes $\Theta(d(u))$ time (worst case)

`adj(`v`)` iteration $O(1)$ per neighbor

$\Theta(n + m)$ (words of) space for any graph (≪ $\Theta(n^2)$ bits for moderate m)

$\rightsquigarrow$ de-facto standard for graph algorithms

Graph Types and Representations

- Note that adj matrix and lists for undirected graphs effectively are representation of directed graph with directed edges both ways
  - conceptually still important to distinguish!
- multigraphs, loops, edge weights all naturally supported in adj lists
  - good if we allow and use them
  - but requires explicit checks to enforce simple / loopfree / bidirectional!
- we focus on **static graphs**
  dynamically changing graphs much harder to handle

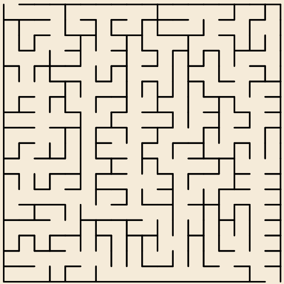
9.3 Graph Traversal

Generic Graph Traversal

- Plethora of graph algorithms can be expressed as a systematic exploration of a graph

visiting all nodes & edges

  - depth-first search, breadth-first search
  - connected components
  - detecting cycles
  - topological sorting
  - Hierholzer's algorithm for Euler walks
  - strong components
  - testing bipartiteness
  - Dijkstra's algorithm
  - Prim's algorithm
  - Lex-BFS for perfect elimination orders of chordal graphs
  - ...



$\leadsto$ Formulate generic traversal algorithm

  - first in abstract terms to argue about correctness
  - then again for concrete instance with efficient data structures

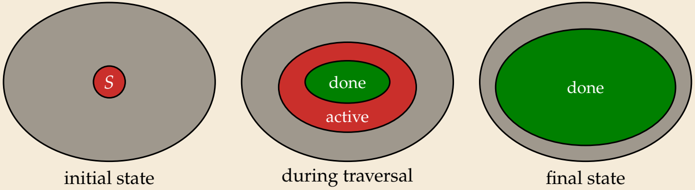
Tricolor Graph Traversal

Tricolor Graph Search:

- maintain vertices in 3 (dynamic) sets
  - **Gray: unseen vertices**
    The traversal has not reached these vertices so far.
  - **Green: done vertices** (a.k.a. visited vertices)
    These vertices have been visited and all their edges have been explored already.
  - **Red: active vertices** (a.k.a. frontier („Rand“) of traversal)
    All others, i. e., vertices that have been reached and some unexplored edges remain; initially some selected start vertices S.

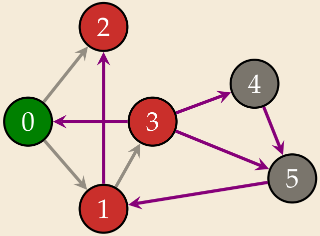
Invariant:
No edges from ***done*** to *unseen* vertices

- (implicitly) maintain status of each edge
  - **not yet used**
  - **used edge**
- Vertices "want" to turn green.



Generic Tricolor Graph Traversal – Code

```
procedure genericGraphTraversal(G, S):
    // (di)graph G = (V, E) and start vertices S ⊆ V
    C[0..n) := unseen // Color array, all cells initialized to unseen
    for s ∈ S do C[s] := active end for
    unusedEdges := E
    while ∃v : C[v] == active
        v := nextActiveVertex() // Freedom 1: Which frontier vertex?
        if ∄vw ∈ unusedEdges // no more edges from v ⇝ done with v
            C[v] := done
        else
            w := nextUnusedEdge(v) // Freedom 2: Which of its edges?
            if C[w] == unseen
                C[w] := active
            end if
            unusedEdges.remove(vw)
        end if
    end while
```



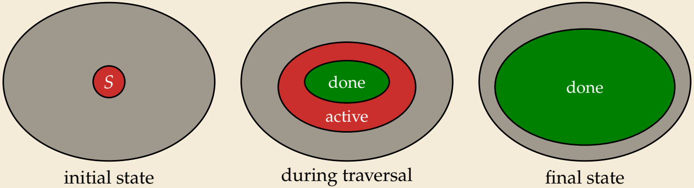
Invariant:
No edges from *done* to *unseen* vertices

- Implementations of nextActiveVertex() and nextUnusedEdge(v) depends on (and defines!) specific traversal-based graph algorithms

Generic Reachability

- Any choices nextActiveVertex() and nextUnusedEdge(v) suffice to find exactly the vertices reachable from S in ***done***

- **Invariant:**
  1. No edges from ***done*** to ***unseen*** vertices
  2. For every ***done*** or ***active*** vertex v, there exists a path from $s \in S$ to v.



$\rightsquigarrow$ in final state:

- $v \in$ ***done*** $\rightsquigarrow$ path from S $\rightsquigarrow$ reachable from S
- $v \in$ ***unseen*** $\rightsquigarrow$ not reachable from ***done*** $\supseteq S$ $\rightsquigarrow$ not reachable from S

Data Structures for Frontier

- We need efficient support for
  - test $\exists v : C[v] = \textbf{\textit{active}}$, nextActiveVertex()
  - test $\exists vw \in \mathit{unusedEdges}$, nextUnusedEdge($v$)
  - $\mathit{unusedEdges}$.remove(vw)
- Typical solution maintains **bag** "*frontier*" of *pairs* (v, i) where $v \in V$ and i is an **iterator** in `adj`$[v]$
  - $\mathit{unusedEdges}$ represented implicitly: edge used iff previously returned by i
    $\rightsquigarrow$ don't need $\mathit{unusedEdges}$.remove(vw)
  - Implement $\exists v : C[v] = \textbf{\textit{active}}$ via *frontier*.isEmpty()
  - Implement $\exists vw \in \mathit{unusedEdges}$ via i.hasNext() assuming $(v, i) \in \mathit{frontier}$
  - Implement nextUnusedEdge(v) via i.next() assuming $(v, i) \in \mathit{frontier}$

$\rightsquigarrow$ all operations apart from nextActiveVertex() in $O(1)$ time

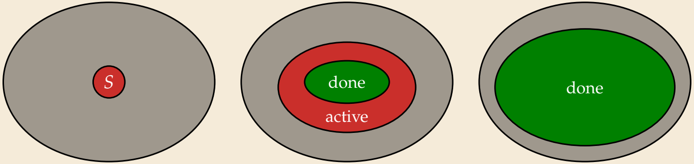
$\rightsquigarrow$ *frontier* requires $O(n)$ extra space

9.4 BFS and DFS

Breadth-First Search

- Maintain *frontier* in a **queue** (FIFO: first in, first out)
- **Invariant:**
  1. No edges from done to unseen vertices
  2. All ***done*** or ***active*** vertices are reached via a **shortest path** from S

     fewest edges
  3. Vertices enter and leave *frontier* in order of increasing distance from S



initial state during traversal final state

⇝ in final state, we reach all reachable vertices via shortest paths

- To preserve that knowledge, we collect extra information during traversal
  - $parent[v]$ stores predecessor on path from S via which v was reached
  - $distFromS[v]$ stores the length of this path

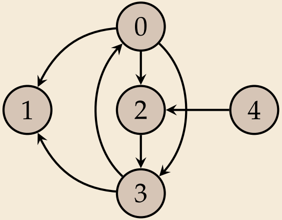
Breadth-First Search – Code

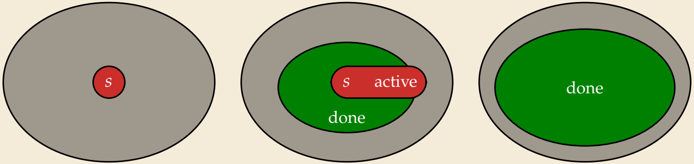
```
procedure bfs(G, S):
    // (di)graph G = (V, E) and start vertices S ⊆ V
    C[0..n) := unseen // New array initialized to all unseen
    frontier := new Queue;
    parent[0..n) := NOT_VISITED;   distFromS[0..n) := ∞
    for s ∈ S
        parent[s] := NONE;   distFromS[s] := 0
        C[s] := active;   frontier.enqueue((s, G.adj[s].iterator()))
    end for
    while ¬frontier.isEmpty()
        (v, i) := frontier.peek()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;   frontier.dequeue()
        else
            w := i.next() // Advance i in adj[v]
            if C[w] == unseen
                parent[w] := v;   distFromS[w] := distFromS[v] + 1
                C[w] := active;   frontier.enqueue((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

- *parent* stores a *shortest-path tree/forest*
- can retrieve shortest path to v from some vertex $s \in S$ (backwards) by following $parent[v]$ iteratively
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

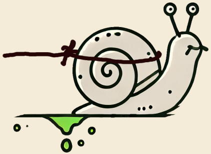
Depth-First Search

- Maintain *frontier* in a **stack** (LIFO: last in, first out)
  - only consider $S = \{s\}$
  - usual mode of operation: call `dfs`(v) for all *unseen* v, for $v = 0, \ldots, n-1$
- **Invariant:**
  1. No edges from done to unseen vertices
  2. All *done* or *active* vertices are reached via a path from s
  3. The *active* vertices form a single **path** from s





initial state | during traversal | final state



Depth-First Search – Code

```
procedure dfsTraversal(G):
    C[0..n) := unseen
    for v := 0, . . . , n − 1
        if C[v] == unseen
            dfs(G, v)

procedure dfs(G, s):
    frontier := new Stack;
    C[s] := active;  frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;  frontier.pop();  postorderVisit(v)
        else
            w := i.next();  visitEdge(vw)
            if C[w] == unseen
                preorderVisit(w)
                C[w] := active;  frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

- define *hooks* to implement further operations
  - preorder: visit v when made ***active*** (start of $T(v)$)
  - postorder: visit v when marked ***done*** (end of $T(v)$)
  - visitEdge: do something for every edge
- if needed, can store DFS forest via *parent* array
- running time $\Theta(n + m)$
- extra space $\Theta(n)$

Simple DFS Application: Connected Components

- In an undirected graph, find all *connected components*.
  - **Given:** simple undirected $G = (V, E)$
  - **Goal:** assign component ids $CC[0..n)$, s.t. $CC[v] = CC[u]$ iff $\exists$ path from v to u

```
procedure connectedComponents(G):
    // undirected graph G = (V, E) with V = [0..n)
    C[0..n) := unseen
    CC[0..n) := NONE
    id := 0
    for v := 0, ..., n − 1
        if C[v] == unseen
            dfs(G, v)
            id := id + 1
    return CC

procedure preorderVisit(v):
    CC[v] := id
```

```
// same as before
procedure dfs(G, s):
    frontier := new Stack;
    C[s] := active; frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done; frontier.pop()
            postorderVisit(v)
        else
            w := i.next(); visitEdge(vw)
            if C[w] == unseen
                preorderVisit(w)
                C[w] := active
                frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

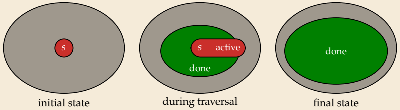
Dijkstra's Algorithm & Prim's Algorithm

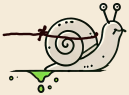
- On edge-weighted graphs, we can use tricolor traversal with a *priority queue* as *frontier*
- Dijkstra's Algorithm for shortest paths from s in digraphs with weakly positive edge weights
  - priority of vertex v = length of shortest path known so far from s to v
- Prim's Algorithm for finding a minimum spanning tree
  - priority of vertex v = weight of cheapest edge connecting v to current tree

$\rightsquigarrow$ Detailed discussion in Unit 11

9.5 Advanced Uses of DFS

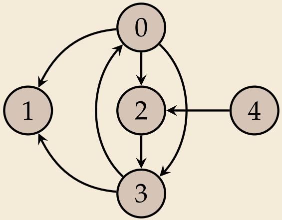
Properties of DFS



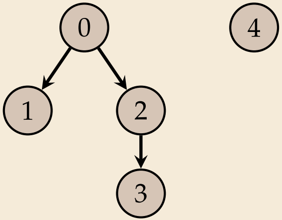


- **Recall DFS Invariant 3:**
  The *active* vertices form a single **path** from s

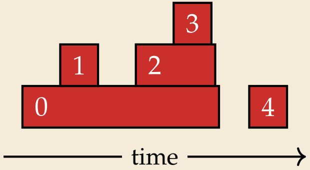
input graph G



DFS forest



stack over time



$\rightsquigarrow$ Each vertex v spends *time interval* $T(v)$ as *active* vertex

1. *frontier* is stack $\rightsquigarrow$ $\{T(v) : v \in V\}$ forms *laminar set family*: (“disjoint or contained”)
either $T(v) \cap T(w) = \emptyset$ or $T(v) \subseteq T(w)$ or $T(v) \supseteq T(w)$

2. Parenthesis Theorem: $T(v) \supseteq T(w)$ **iff** v is ancestor of w in DFS tree

‘$\Rightarrow$’ during $T(v)$, all discovered vertices become descendants of v

‘$\Leftarrow$’ $T(v)$ covers v’s entire subtree, which contains w’s subtree

Properties of DFS – Unseen-Path Theorem

- **Unseen-Path Theorem:** In a DFS forest of a (di)graph G, w is a descendant of v **iff** at the time of `preorderVisit`(v), there is a path from v to w using only *unseen* vertices.

'⇒' If w is a descendant of v, $T(w) \subseteq T(v)$ by the Parenthesis Theorem.
Hence the path from v to w in the DFS tree consists (at time of `preorderVisit`(v)) of solely *unseen* vertices.

'⇐' Suppose towards a contradiction that there was a w with an *unseen* path $p[0..\ell]$ with $p[0] = v$ and $p[\ell] = w$, but w is not a descendant of v. W.l.o.g. let w be a first such vertex, i. e., $p[0], \ldots, p[\ell - 1] = u$ are descendants of v.
So $T(u) \subset T(v) \quad (*)$.
Upon processing u, we will discover edge uw, so whether or not w is already ***done*** at this point, w will be marked ***done*** before u. Hence $\max T(w) \le \max T(u)$.
With $(*)$, we obtain $\min T(v) \le \min T(u) \le \max T(w) \le \max T(u)$, so by laminarity, $T(w) \subset T(u) \subset T(v)$ and w is a descendant of v ↯.

Topological Sorting & Cycle Detection

- **Application:** Given a set of tasks with precedence constraints of the form "a must be done before b", can we find a legal ordering for all tasks?

  $\rightsquigarrow$ Model as directed graph!
  - tasks are the vertices V
  - add an edge (a, b) when a must be done before b

- **Definition:** $R[0..n)$ is a *topological (order) ranking* of digraph $G = (V, E)$ if $\forall (u, v) \in E : R[u] < R[v]$

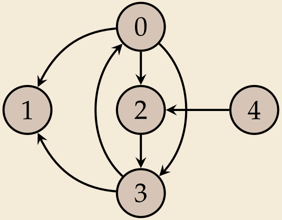
- **Lemma DAG iff topo:**
  A directed graph G has a topological ranking **iff** it does not contain a directed cycle.

- **Topological Sorting**
  - **Given:** simple digraph $G = (V, E)$
  - **Goal:** Compute topological ranking of vertices $R[0..n)$
    or output a directed cycle in G.

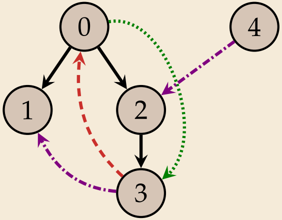
- Amazingly, can do all with one pass of DFS!

DFS Edge Types

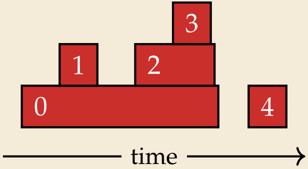
input digraph G



DFS forest



stack over time



- During DFS traversal, an edge vw has one of these 4 types: **example:**
  1. **tree edge:** $\longrightarrow$ $w \in$ *unseen* $\rightsquigarrow$ vw part of DFS forest. $(0,1), (0,2), (2,3)$
  2. **back edges:** $w \in$ ***active***; $\rightsquigarrow$ w points to ancestor of v. $(3,0)$
  3. **forward edges*:** $w \in$ ***done*** $\wedge$ w is descendant of v in DFS tree. $(0,3)$
  4. **cross edges*:** $w \in$ ***done*** $\wedge$ w is not descendant of v. $(3,0)$

  *only possible in *directed* graphs

Cycle Detection

If G contains a directed cycle, DFS will find a directed cycle:

- any back edge implies a cycle:
  - DFS visits an edge (v, w) where $w \in$ ***active***, w is already on the stack
  - $\rightsquigarrow$ DFS tree contains path $w \rightsquigarrow v$ and we have edge $v \to w$.
- conversely any cycle $C[0..k]$ once reached must have some back edge or cross edge (tree and forward edges go from smaller to larger preorder index)
  - cannot be a cross edge since cycle is strongly connected
    all cycle vertices must be descendants of first reached cycle vertex
  - $\rightsquigarrow$ cycle contributes a back edge

DFS Postorder Implementation

```
procedure dfsPostorder(G):
    C[0..n) := unseen
    P[0..n) := NONE;  r := 0
    parent[0..n) := NONE
    cycle := NONE
    for v := 0, ..., n − 1
        if C[v] == unseen
            dfs(G, v)
    return (P, cycle)

procedure postorderVisit(v):
    P[v] := r;  r := r + 1

procedure visitEdge(vw):
    if C[w] == active
        if cycle ≠ NONE return
        while v ≠ w
            cycle.append(v)
            v := parent[v]
        cycle.append(v)
```

```
// dfs is as in CC but with parent
procedure dfs(G, s):
    frontier := new Stack;
    parent[s] := NONE;
    C[s] := active;  frontier.push((s, G.adj[s].iterator()))
    while ¬frontier.isEmpty()
        (v, i) := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            C[v] := done;  frontier.pop()
            postorderVisit(v)
        else
            w := i.next() // Advance i in adj[v]
            visitEdge(vw)
            if C[w] == unseen
                parent[w] := v;
                preorderVisit(w)
                C[w] := active;  frontier.push((w, G.adj[w].iterator()))
            end if
        end if
    end while
```

DFS Postorder & Topological Sort

- **DFS Postorder**: The DFS postorder numbers is a numbering $P[0..n)$ of V such that $P[v] = r$ iff exactly r vertices reached state ***done*** before v in a DFS.

- **Lemma rev postorder:**
  directed acyclic graph
  Let G be a simple, connected DAG and $R[0..n)$ a *reverse DFS postorder* of G, i. e., $R[v] = n - 1 - P[v]$ for a DFS postorder $P[0..n)$. Then R is a topological ranking of G.

- **Invariant:** If $v \in$ ***done*** and $(v, w) \in E$ then $w \in$ ***done*** and $R[v] < R[w]$.
  - initially true (***done*** $= \emptyset$)
  - upon `postorderVisit`(v), all outgoing edges vw lead to $w \in$ ***done*** (Parenthesis Theorem)

Topological Sorting & Cycle Detection – Summary

- Putting everything together we obtain topological sorting
  - can produce either the *ranking* or the *sequence of vertices* in topological order, whatever is more convenient

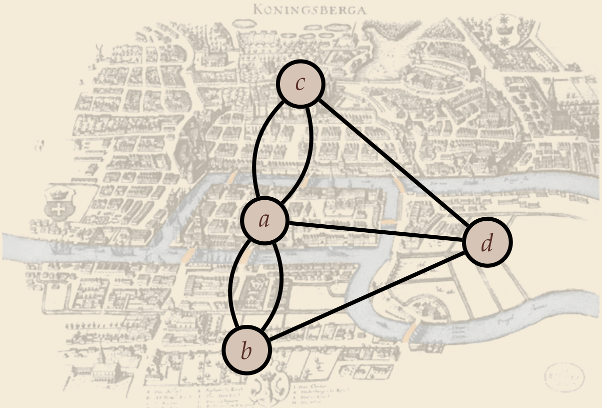
```
procedure topologicalRanking(P):
    (P[0..n), cycle) := dfsPostorder(G)
    if cycle ≠ NULL
        return NOT_A_DAG
    R[0..n) := NONE
    for v := 0, . . . , n − 1
        R[v] = n − 1 − P[v]
    return R
```

```
procedure topologicalSort(P):
    (P[0..n), cycle) := dfsPostorder(G)
    if c ≠ NULL
        return NOT_A_DAG
    S[0..n) := NONE
    for v := 0, . . . , n − 1
        S[n − 1 − P[v]] := v
    return S
```

- $\Theta(n + m)$ time
- $\Theta(n)$ extra space

Euler Cycles

Euler Walk: Walk using every edge in $G = (V, E)$ exactly once.

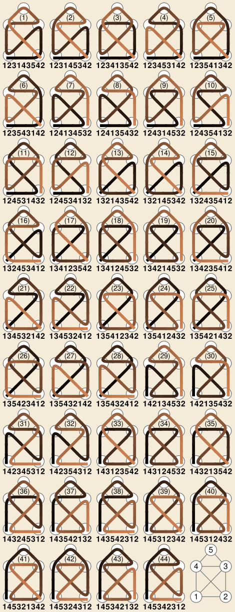


Euler's Theorem:

Euler walk exists iff G connected and 0 or 2 vertices have odd degree.

'$\Rightarrow$' trivial (need to enter and exit intermediate vertices equally often)

'$\Leftarrow$' Following algorithm *constructs* Euler walk under this assumption



Euler Cycles – Hierholzer's Algorithm

- use an *edge-centric DFS*
  - We mark *edges* (not vertices)
  - ⇝ stack = **edge-simple walk**
  - We remember iterator i globally per v to resume traversal

```
procedure eulerWalk(G):
    // Assume G = (V, E) is connected (multi)graph
    V_odd := {v ∈ V : d(v) odd}
    if |V_odd| ∉ {0,2} return NOT_EULERIAN
    if V_odd = {x, y} then s := x else s := 0
    euler[0..m) := NONE;   j := m − 1
    visited[0..n, 0..n) := false // mark edges as visited
    for v := 0, ..., n − 1
        // globally remember next unexplored edge
        nextEdge[v] := G.adj[w].iterator())
    edgeDFS(s)
    return euler
```

```
procedure edgeDFS(s):
    frontier := new Stack;
    frontier.push(s)
    while ¬frontier.isEmpty()
        v := frontier.top()
        if ¬i.hasNext() // v has no unused edge
            frontier.pop()
            if ¬frontier.isEmpty()
                // assign edge leading here largest free index
                euler[j] := (frontier.top(), v);   j := j − 1
            end if
        else
            w := i.next()
            if ¬visited[v, w]
                visited[v, w] := true
                visited[w, v] := true
                frontier.push(w)
            end if
        end if
    end while
```

Strong Components

- **Given:** digraph $G = (V, E)$
- **Goal:** component ids $SCC[0..n)$, s.t. $SCC[v] = SCC[u]$ iff $\exists$ directed path from v to u

strongly connected component

- *Component DAG G^{SCC}:* contract SCCs intro single vertices
  $V(G^{SCC}) = \{C_1, \ldots, C_k\}$ with $C_1 \dot\cup \cdots \dot\cup C_k = V$;
  name by smallest vertex s.t. $i \le j$ iff $\min C_i \le \min C_j$
  - can't have cycles (↯ maximality of SCC)
  - $\rightsquigarrow$ component DAG has a topological order $R^{SCC}[1..k]$

If we call `dfs` on any v in the **last** SCC C, it will discover all vertices in C, and only those!
(any edges between components lead *into* C by topological order)

And we can iterate this backwards through any topological order to get all SCCs!

Can we efficiently find the topological order of G^{SCC}?
Without knowing the components to start with??

Amazingly, yes.

Component Graph DFS

- Suppose we run `dfsTraversal` on G.
  - ⇝ We can extend time intervals to SCCs: $T(C_i) := \bigcup_{v \in C_i} T(v)$
  - ⇝ $T(C_i) = T(v_i)$ for $v_i \in C_i$ the first vertex to be explored in a DFS on G (by Unseen Path & Parenthesis Thms)

⇝ DFS on G produces same $T(C_i)$ (up to time scaling) as DFS on G^{SCC}!

⇝ reverse DFS postorder on G gives same relative order to $v_1, \ldots, v_k$ as reverse DFS postorder on G^{SCC} gives as relative order to $C_1, \ldots, C_k$

We need **reverse** topological order on G^{SCC}, e. g., ***reversed*** *reverse DFS postorder*

- If we had the actual reverse DFS postorder on G^{SCC}, could just reverse again!
- But we only have reverse DFS postorder $S[0..n)$ on G!
- ϟ Reversing here would change v_i, i. e., which vertices of an SCC we see first

Kosaraju-Sharir's Algorithm

- **Recall:** Want $\texttt{reverse}\big(\texttt{topologicalRanking}(G^{\text{SCC}})\big)$
- **Transpose/Reverse Graph of $G = (V, E)$:** $G^T = (V, E^T)$ where $E^T = \{wv : vw \in E\}$

  Note: A adj matrix of G $\rightsquigarrow$ A^T adj matrix of G^T
- For any DAG, we obtain a reverse topological order from reversing all edges: $\texttt{topologicalSort}(G^T)$

  If we reverse *iteration order* in dfsTraversal, we get $\texttt{reverse}(\texttt{topologicalSort}(G)) = \texttt{topologicalSort}(G^T)$
- Observation: $\left(G^T\right)^{\text{SCC}} = \left(G^{\text{SCC}}\right)^T$
  - strong components not affected by edge reversals
- Want: $\texttt{reverse}\big(\texttt{topologicalRanking}(G^{\text{SCC}})\big)$ (any ranking works, need not be reverse DFS postorder)

$\rightsquigarrow$ Get it from: $\texttt{topologicalRanking}\big((G^{\text{SCC}})^T\big) = \texttt{topologicalRanking}\big((G^T)^{\text{SCC}}\big)$

$\rightsquigarrow$ Get that as induced ranking on $v_1, \ldots, v_k$ from reverse $\texttt{dfsPostorder}(G^T)$

Kosaraju-Sharir's Algorithm – Code

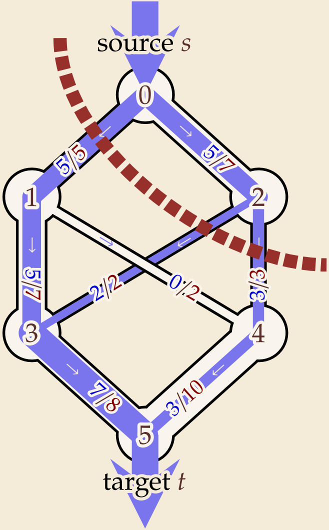
```
procedure strongComponents(G):
    // directed graph G = (V, E) with V = [0..n)
    G^T = (V, {wv : vw ∈ E})
    P[0..n) := dfsPostorder(G^T) // postorder numbers
    for v ∈ V do S[P[v]] := v end for // postorder sequence
    // Rest like connectedComponents (with permuted vertices)
    C[0..n) := unseen
    SCC[0..n) := NONE
    id := 0
    for j := n − 1, . . . , 0 // reverse postorder seq
        v := S[j]
        if C[v] == unseen
            dfs(G, v)
            id := id + 1
    return SCC

procedure preorderVisit(v):
    SCC[v] := id
```

- correctness follows from our discussion
- ordering of SCCs follows reverse topological sort of G^{SCC}
  - some implementations reverse G for 2nd DFS, not 1st
  - ⇝ output in (forward) topological order
  - but derivation more natural this way?
- as all our traversals: $\Theta(n + m)$ time, $\Theta(n)$ extra space

9.6 Network flows

Networks and Flows – Informal



Informally, imagine a network of water pipes.

- Water can flow through the pipes up to a flow capacity limit (up to $c(e)$ liters per second, say).
- There's infinite water pressing into the source s and infinite drain capacity at the sink / target t
- At all other junctions, inflow = outflow (no leakage)

⇝ How much water can flow through the network?

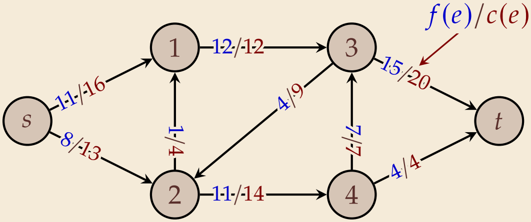
In this example:

- not more than $5 + 2 + 3 = 10$ units of flow out of $\{0, 2\}$ possible
- ⇝ not more than 10 units out of s possible
- ⇝ shown flow is maximal

Remainder of this unit: general version of above (+ efficient algorithms)

Networks and Flows – Definitions

- *s-t-(flow) network*:
  - **simple**, **directed**, **connected** graph $G = (V, E)$, no antiparallel edges ($vw \in E \rightsquigarrow wv \notin E$)
    for notational convenience only
  - *edge capacities* $c : E \to \mathbb{R}_{\geq 0}$
  - distinguished vertices: *source* $s \in V$, target/*sink* $t \in V$
- *(network) flow (in G)*: $f : E \to \mathbb{R}_{\geq 0}$
- flow f is *feasible* if it satisfies
  notational convenience: set $f(vw) = c(vw) = 0$ for $vw \notin E$
  - *capacity constraints:* $\forall v, w \in V \ : \ 0 \leq f(vw) \leq c(vw)$
  - *flow conservation:* $\forall v \in V \setminus \{s, t\} \ : \ \sum_{w \in V} f(w, v) = \sum_{w \in V} f(v, w)$
- *value* $|f|$ of flow f: $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$



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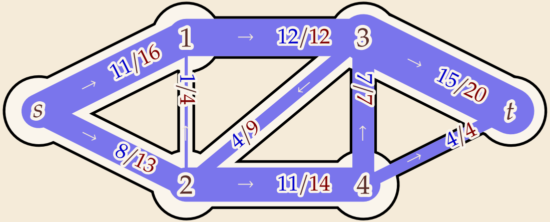
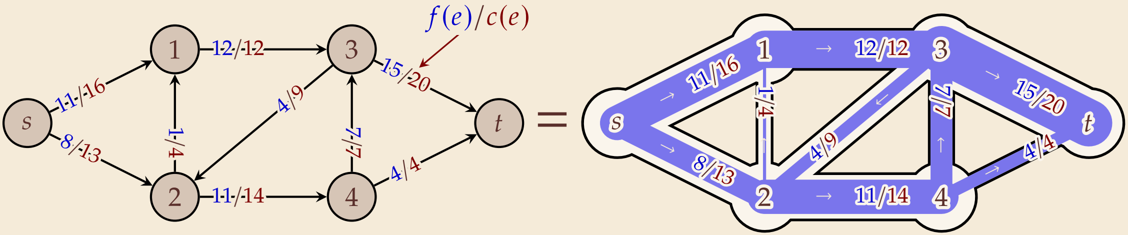


Max-Flow Problem



- **Maximum-Flow Problem:**
  - **Given:** s-t-flow network
  - **Goal:** Find feasible flow f^* with maximum $|f^*|$ among all feasible flows

- $\mathbb{N}$ vs $\mathbb{R}$
  - We focus on integral capacities here $\rightsquigarrow$ can restrict ourselves to integral flows
    as we will see
  - but: ideally want algorithms that work with arbitrary real numbers, too

Multiple Sources & Sinks, Antiparallel Edges

- Some of the restrictions can be generalized easily.
- We forbid **loops** and **antiparallel** edges.
  - The presented algorithms actually work fine with both!
  - but proofs are cleaner to write without them
  - also: can always remove loops and (anti)parallel edges by adding a new vertex in the middle of the edge

  $\leadsto$ same maximum $|f|$

- We only allow a **single source** and a **single sink**
  - can add a **"supersource"** and **"supersink"** with capacity-∞ edges to all sources resp. sinks

  $\leadsto$ same maximum $|f|$

Reductions

- Apart from directly modeling (data, traffic, etc.) flow, a key reason to study network flows are reductions of other problems

1. **Disjoint Paths**
   - **Given:** Unweighted (di)graph $G = (V, E)$, vertices $s, t \in V$
   - **Goal:** How many edge-disjoint paths are there from s to t?

2. **Assignment Problem, Maximum Bipartite Matching**
   - **Given:** workers $W = \{w_1, \ldots, w_k\}$ tasks $T = \{t_1, \ldots, t_\ell\}$, *qualified-for* relation $Q \subseteq W \times T$
   - **Goal:** Assignment $a : W \to T \cup \{\perp\}$ of workers to tasks such that
     - workers are qualified: $\forall w \in W : a(w) \neq \perp \implies (w, a(w)) \in Q$
     - $|a(W)|$, the number of tasks assigned, is maximized

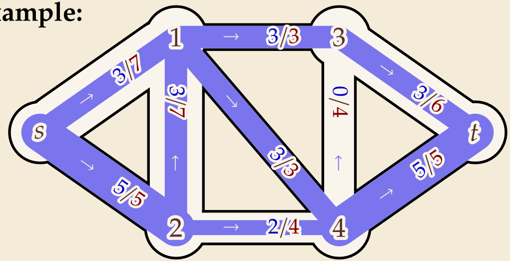
- Both problems can be solved by (in both cases, 1. and 3. are very efficient)
  1. constructing a specific flow network from their input data
  2. computing a maximum flow in that network
  3. "reading off" a solution for the original problem from the max flow

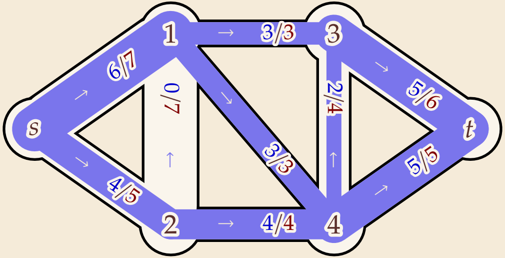
9.7 The Ford-Fulkerson Method

Push Push Push!?

- **Simple Idea:** Iteratively find a path from s to t that we can push more flow over.

Example:





1. Push 3 units of flow over $s \to 2 \to 1 \to 3 \to t$
2. Push 3 units of flow over $s \to 1 \to 4 \to t$
3. Push 2 units of flow over $s \to 2 \to 4 \to t$

$\rightsquigarrow$ Every s-t path now has a saturated edge.

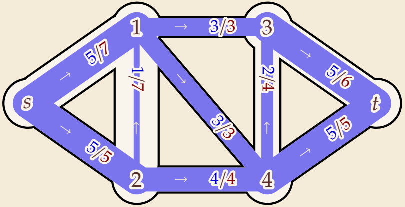
⚠ **But:** resulting flow is **not** optimal!

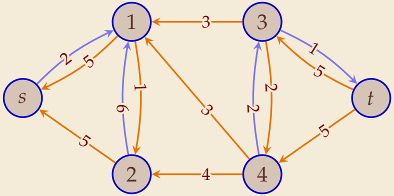
Problem: Cannot undo mistakes. Here: shouldn't have put so much flow on $(1, 2)$ ...

Residual Networks

- Goal: Allow undoing flow (without backtracking)
- *Residual network* G_f: given network $G = (V, E)$ and feasible flow f
  - $G_f = (V, E_f)$ with capacities $c_f(vw) = \begin{cases} c(vw) - f(vw) & vw \in E \quad \textit{// add flow} \\ f(wv) & wv \in E \quad \textit{// revert flow} \\ 0 & \text{else} \end{cases}$

    $E_f = \{vw : c_f(vw) > 0\}$





- *residual flow* f': feasible flow in G_f

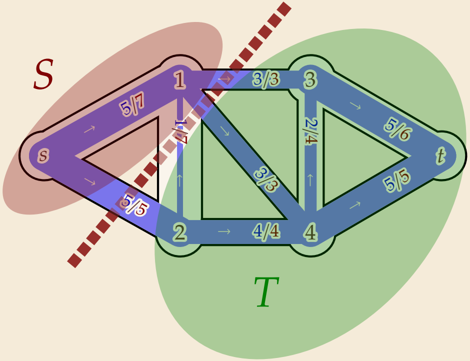
  $(f + f')(vw) = f(vw) + f'(vw) - f'(wv)$

  $\rightsquigarrow$ for any f and residual flow f' in G_f, flow $f + f'$ is a feasible flow in G

- *augmenting path* p: s-t-path G_f particularly simple f'!

Cuts

- **Goal:** Certificate for maximum flows
- *s-t-cut* (S,T): partition $S \dot\cup T = V$, $s \in S$, $t \in T$
  - *net flow* across cut:
    $$f(S,T) = \sum_{v \in S} \sum_{w \in T} \big(f(vw) - f(wv)\big)$$
  - *capacity* of cut:
    $$c(S,T) = \sum_{v \in S} \sum_{w \in T} f(vw)$$



- $f(S,T) = 5 + 3 + 3 - 1 = 10$
- $c(S,T) = 5 + 3 + 3 = 11$

- **Lemma:** For any cut (S,T), we have $f(S,T) = |f|$.
  (flow conservation!)
- **Corollary:** $|f| \le c(S,T)$ for any s-t-cut (S,T)

The Max-Flow Min-Cut Theorem

- **Max-Flow Min-Cut Theorem:**
  Let f be a feasible flow in s-t-network $G = (V, E)$. Then the following conditions are equivalent:
  1. $|f| = c(S, T)$ for some cut (S, T) of G.
  2. f is a maximum flow in G
  3. The residual network G_f has no augmenting path.

Generic Ford-Fulkerson Method

```
procedure genericFordFulkerson(G = (V, E), s, t, c):
    // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
    for vw ∈ E do f(vw) := 0 end for
    while ∃ path p from s to t in G_f // Freedom: Which augmenting path?
        Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
        for e ∈ p
            if e ∈ E // forward edge
                f(e) := f(e) + Δ
            else // backward edge
                f(e) := f(e) − Δ
    return f
```

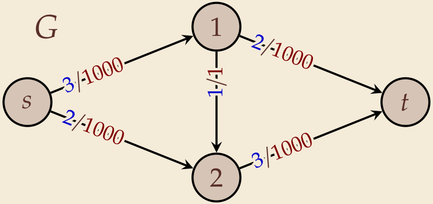
- Returned flow is a maximum flow f^* (Max-Flow Min-Cut Theorem)
- If $c : E \to \mathbb{N}_0$, also $f : E \to \mathbb{N}_0$: For all $v, w \in V$ holds:
  - initially $f(vw) = 0 \in \mathbb{N}_0$
  - $c_f(vw)$ is difference of $c(vw) \in \mathbb{N}_0$ and $f(vw) \in \mathbb{N}_0$
  - Δ equal to some $c_f(v'w') \in \mathbb{N}_{\geq 1}$ (E_f contains only non-zero capacity edges!)

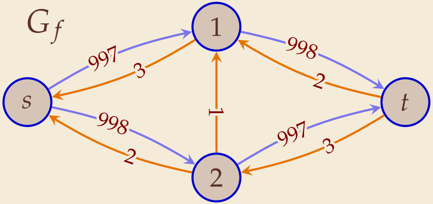
  $\rightsquigarrow$ new flow $f(vw) \pm \Delta \in \mathbb{N}_0$

$\rightsquigarrow$ For integral capacities, always terminate after $\leq |f^*|$ iterations

Bad Example

- Unfortunately, we might also take $|f^*|$ iterations!

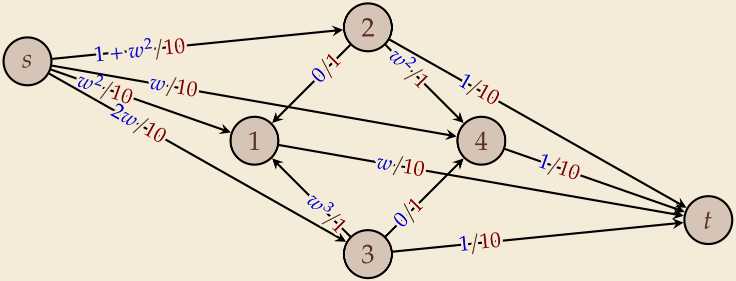




- (2 iterations with smarter augmenting paths would have sufficed here)

A Very Bad Example

- for irrational flows, might not even terminate
- example network with irrational initial flow
- $w = \varphi - 1 = (\sqrt{5} - 1)/2 \approx 0.618 \quad\rightsquigarrow\quad 1 - w = w^2 \approx 0.382$



- after 2 paths, situation in 1-2-3-4 restored (rotated), but flows multiplied by w

$\rightsquigarrow$ augmenting paths have capacities $w, w, w^2, w^2, w^3, w^3 \ldots$

$\rightsquigarrow$ never terminate, never exceed $|f| \geq 5$

9.8 The Edmonds-Karp Algorithm

Edmonds-Karp

- It turns out, many ways to choose augmenting paths systematically work fine
- Edmonds & Karp: take a shortest path (in #edges)

```
procedure EdmondsKarp(G = (V, E), s, t, c):
    // G is a flow network with source s ∈ V, sink t ∈ V and capacities c : E → ℝ≥0
    for vw ∈ E do f(vw) := 0 end for
    while true
        bfs(G_f, {s})
        if distFrom[t] == ∞ return f
        else p := pathTo(t)
        Δ := min{c_f(e) : e ∈ p} // bottleneck capacity
        for e ∈ p
            if e ∈ E // forward edge
                f(e) := f(e) + Δ
            else // backward edge
                f(e) := f(e) − Δ
    end while
```

Edmonds-Karp – Analysis

- **Theorem:** The Edmonds-Karp algorithm terminates after $O(nm)$ iterations with a maximum flow. The total running time is in $O(nm^2)$.

- *Proof Plan:*
  - every augmenting path has a *critical* edge vw contributing the bottleneck capacity
  - we will show:
    (1) distances of vertices from s in G_f weakly increase over time
    (2) before vw can be a *critical* edge *again*, v's distance increases by at least 2
  - $\rightsquigarrow$ each edge vw is critical for at most $n/2$ augmenting paths (v's distance $\in [1..n-2]$)
  - $\rightsquigarrow$ $O(nm)$ augmenting paths
  - each iteration runs one BFS, which costs $O(n+m) = O(m)$ times since G is connected.

- **Notation:**
  - Write $f_0, f_1, \ldots$ for values of f during iterations of while loop
  - $\rightsquigarrow$ G_{f_i} residual network after ith augmentation
  - Write $\delta_i(v)$ for shortest-path distance from s to v in G_{f_i}

Edmonds-Karp – Analysis [2]

- **EK Monotonicity Lemma:** For all i and $v \in V$, we have $\delta_{i+1}(v) \geq \delta_i(v)$.

- f_i: flow after ith augmentation
- $\delta_i(v)$ distance from s to v in G_{f_i}

Proof:

- by induction over k, the value of $\delta_i(v)$
- IB: $k = 0$: only $v = s$ possible; $\delta_{i+1}(s) = 0 \geq 0 = \delta_i(s)$ ✓
- IH: Assume the claim is true for all shortest paths up to length k
- IS: Suppose $\delta_{i+1}(v) = k + 1$.

  $\rightsquigarrow$ $\exists$ shortest path $p[0..k+1]$ in $G_{f_{i+1}}$ with $p[0] = s$ and $p[k+1] = v$.

  $\rightsquigarrow$ For $w = p[k]$, $p[0..k]$ is a shortest path from s to w $\rightsquigarrow$ $k = \delta_{i+1}(w) \underset{\text{IH}}{\geq} \delta_i(w)$

  - Case 1: $wv \in E_{f_i}$ $\rightsquigarrow$ $\delta_i(v) \leq \delta_i(w) + 1$
  - Case 2: $wv \notin E_{f_i}$ $\rightsquigarrow$ reverse edge vw in ith augmenting path, a shortest s-t-path

    $\rightsquigarrow$ $\delta_i(v) = \delta_i(w) - 1 \leq \delta_i(w) + 1$
  - in both cases: $\delta_{i+1}(v) = \delta_{i+1}(w) + 1 \geq \delta_i(w) + 1 \geq \delta_i(v)$

Edmonds-Karp – Analysis [3]

- **Critical Distance Lemma:** When critical edge vw becomes a critical again, $\delta(v)$ has increase by at least 2.

Proof:

- Suppose vw is critical in ith iteration $\rightsquigarrow$ lies on shortest path

$\rightsquigarrow$ $\delta_i(w) = \delta(i)(v) + 1$

- before vw reappears in G_f, need to have had wv in augmenting path;
  say this first happens in iteration $j > i$ $\rightsquigarrow$ $\delta_j(v) = \delta_j(w) + 1$
- by EK Monotonicity Lemma:
  $\delta_j(v) \;=\; \delta_j(w) + 1 \;\geq\; \delta_i(w) + 1 \;=\; \delta_i(v) + 2$

This concludes the proof of the theorem.

Maximum Flow – Discussion

- 👍 Edmonds-Karp is a robust choice
- 👍 easy to implement (see Sedgewick Wayne for an elegant Java version!)
- 👎 worst-case time $O(n^5)$ for dense graphs quickly prohibitive
  - but: worst-case results typically overly pessimistic
  - other choices of augmenting flows possible
  - in practice: push-relabel methods often faster

- 2022 theory breakthrough: almost linear(!) $O(m^{1+o(1)})$ time max flow algorithm
  Chen, Kyng, Liu, Peng, Gutenberg & Sachdeva, FOCS 2022

- max-flow min-cut theorem is a special case of LP duality

- can also solve generalization of min-cost flows
  - each edge vw has a cost $a(vw)$
  - cost of a flow f: $\sum a(vw) \cdot f(vw)$
  - demand d at sink becomes part of constraints: $|f| \geq d$