12 Dynamic Programming

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Prof. Dr. Sebastian Wild

CS566 (Wintersemester 2025–26)
Philipps-Universität Marburg

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Learning Outcomes

Unit 12: ***Dynamic Programming***

1. Be able to apply the DP paradigm to solve new problems.

Outline

12 Dynamic Programming

12.1 Elements of Dynamic Programming

Introduction

- *Dynamic Programming (DP)* is a powerful algorithm **design pattern** for exact solutions to **optimization** problems

  applicable to many problems

- Some commonalities with Greedy Algorithms, but with an element of brute force added in

  DP = "careful brute force" (Erik Demaine)

- often yields polynomial time, but usually not linear time algorithms

- for many problems the *only* way we know to build efficient algorithms

- **Naming fun:** The term "dynamic programming", due to Richard Bellman from around 1953, does not refer to computer programming; rather to a program (= plan, schedule) changing with time. It seems to have been at least partly marketing babble devoid of technical meaning . . .

Plan of the Unit

1. Abstract steps of DP (briefly)
2. Details on a concrete example (*matrix chain multiplication*)
3. More examples!

The 6 Steps of Dynamic Programming

1. Define **subproblems** (and relate to original problem)
2. **Guess** (part of solution) $\rightsquigarrow$ local brute force
3. Set up **DP recurrence** (for quality of solution)
4. Recursive implementation with **Memoization**
5. Bottom-up **table filling** (topological sort of subproblem dependency graph)
6. **Backtracing** to reconstruct optimal solution

- Steps 1–3 require insight / creativity / intuition;
  Steps 4–6 are mostly automatic / same each time

$\rightsquigarrow$ Correctness proof usually at level of DP recurrence

👍 running time too! worst case time = #subproblems · time to find single best guess

When does DP (not) help?

- *No Silver Bullet*
  DP is the most widely applicable design technique, but can't *always* be applied

1. Vitally important for DP to be correct:

Bellman's Optimality Criterion

For a *correctly guessed* fixed part of the solution,
***any* optimal solution to the corresponding subproblems**
must yield an *optimal solution* to the overall problem (once combined).

at most polynomial in n

2. Also, the total **number of different subproblems** should be *"small"*
   (DP potentially still works correctly otherwise, but won't be *efficient*.)

12.2 DP & Matrix Chain Multiplication

The Matrix-Chain Multiplication Problem

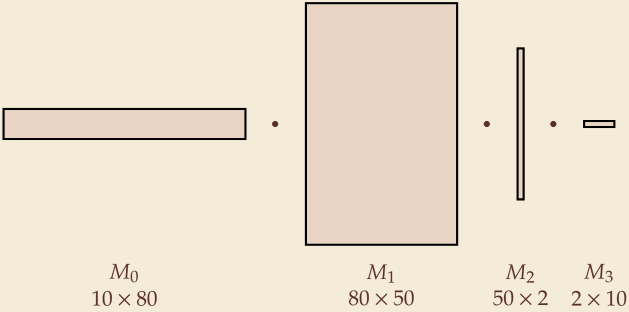
Consider the following exemplary problem

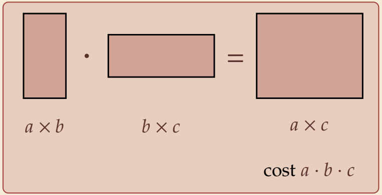
- We have a product $M_0 \cdot M_1 \cdot \cdots \cdot M_{n-1}$ of n matrices to compute
- Since (matrix) multiplication is associative, it can be evaluated in different orders.
- For non-square matrices of different sizes, different order can change costs dramatically
  - Assume elementary matrix multiplication algorithm:

  $\rightsquigarrow$ Multiplying $a \times b$-matrix with $b \times c$ matrix costs $a \cdot b \cdot c$ integer multiplications
- **Given:** Row and column counts $r[0..n)$ and $c[0..n)$ with $r[i+1] = c[i]$ for $i \in [0..n-1)$ (corresponding to matrices $M_0, \ldots, M_{n-1}$ with $M_i \in \mathbb{R}^{r[i] \times c[i]}$)
- **Goal:** parenthesization of the product chain with minimal cost

really a binary tree with n leaves!

Matrix-Chain Multiplication – Example





Parenthesization	Cost (integer multiplications)		
$M_0 \cdot \big(M_1 \cdot (M_2 \cdot M_3)\big)$	$1000 + 40\,000 + 8000$	$=$	$49\,000$
$M_0 \cdot \big((M_1 \cdot M_2) \cdot M_3\big)$	$8000 + 1600 + 8000$	$=$	$17\,600$
$(M_0 \cdot M_1) \cdot (M_2 \cdot M_3)$	$40\,000 + 1000 + 5000$	$=$	$46\,000$
$\big(M_0 \cdot (M_1 \cdot M_2)\big) \cdot M_3$	$8000 + 1600 + 200$	$=$	$9\,800$
$\big((M_0 \cdot M_1) \cdot M_2\big) \cdot M_3$	$40\,000 + 1000 + 200$	$=$	$41\,200$

first or last operation

Greedy fails both ways!

Matrix-Chain Multiplication – How about Brute Force?

If Greedy doesn't give optimal parenthesization, maybe just try all?

- parenthesizations for n matrices $=$ binary trees with n leaves (*evalution trees*)
  $=$ binary trees with $n-1$ (internal) nodes
- How many such trees are there?
  - Let's write $m = n - 1$;
  - $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5$
  - $C_m = \sum_{r=1}^{m} C_{r-1} \cdot C_{m-r} \qquad (m \geq 1)$

    generating functions / combinatorics / guess (OEIS!) & check ...
  - Can show $C_n = \dfrac{1}{n+1}\dbinom{2n}{n} \sim \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{4^n}{n^{3/2}}$

  $\rightsquigarrow$ *exponentially many trees (almost 4^n)* $\qquad C_{20} = 6\,564\,120\,420, \quad C_{30} = 3\,814\,986\,502\,092\,304$

$\rightsquigarrow$ A brute-force approach is utterly hopeless

$\rightsquigarrow$ *Dynamic programming to the rescue!*

Matrix-Chain Multiplication – Step 1: Subproblems

- Key ingredient for DP: Problem allows for recursive formulation
  Need to decide:
  1. What are the **subproblems** to consider?
  2. How can the **original problem** be expressed as subproblem(s)?
- Often requires to solve a more general version of the problem

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

Here:

1. **Subproblems** = Ranges of matrices $[i..j)$ $\quad 0 \le i \le j \le n$
   i. e., optimal parenthesization for each range $M_i, M_{i+1}, \ldots, M_{j-1}$
2. **Original problem** = range $[0..n)$

- **Intuition:**
  - Any subtree in binary multiplication tree covers some range $[i..j)$
    (matrix multiplication is not commutative $\leadsto$ left-right order has to stay)
  - left and right factors of a multiplication don't "see/influence" each other

Matrix-Chain Multiplication – Step 2: Guess

- Usually, any subproblem can be split into smaller subproblems in **several** ways
- Which way to decompose gives best solution not known *a priori*

$\rightsquigarrow$ What do we have to correctly *guess* to solve the problem?

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- Here: **Guess** last multiplication / root of binary tree

$\rightsquigarrow$ index $k \in [i+1\,..\,j)$ so that $[i..j)$ computed with **last** multiplication $\left(M_i \cdot \cdots \cdot M_{k-1}\right) \cdot \left(M_k \cdot \cdots \cdot M_{j-1}\right)$

$\rightsquigarrow$ optimal parenthesization of $M_i, \ldots, M_{k-1}$ and $M_k, \ldots, M_{j-1}$ computed recursively (corresponds to subproblems $[i..k)$ and $[k..j)$)

Matrix-Chain Multiplication – Step 3: DP Recurrence

- With subproblems and guessed part fixed,
  we try to express total **value/cost of solution** *recursively*

$\rightsquigarrow$ *We ignore the actual solution and just compute its cost!*

- Often good to prove correctness at level of recurrence

1. Subproblems
2. Guess!
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- Here: **Recurrence** for $m(i,j)$ = total number of integer multiplications used in best parenthesization of $[i..j)$

$\rightsquigarrow$ Set up recurrence, including any base cases.

$$m(i,j) \;=\; \begin{cases} 0 & \text{if } j - i \le 1 \\ \underset{\text{best } k \text{ chosen by \textit{local brute force}}}{\min}\left\{ \boxed{\overbrace{m(i,k) + m(k,j)}^{\text{recursive cost}} + \overbrace{r[i] \cdot r[k] \cdot c[j-1]}^{\text{cost of last multiplication}}} \;:\; k \in [i+1 \,..\, j) \right\} & \text{otherwise} \end{cases}$$

Matrix-Chain Multiplication – Correctness

Claim: Let $m(i,j)$ for $0 \le i \le j \le n$ be defined by the recurrence

$$m(i,j) \;=\; \begin{cases} 0 & \text{if } j-i \le 1 \\ \min\bigl\{ m(i,k) + m(k,j) + r[i]\cdot r[k]\cdot c[j-1] \;:\; k \in [i+1\,..\,j) \bigr\} & \text{otherwise} \end{cases}$$

Then $m(i,j) \;=\;$ #integer multiplications in best parenthesization of $M_i \cdots M_{j-1}$.

Proof: By induction over $j-i$

- **IB:** When $j-i \le 1$ we have an empty product ($j=i$) or a single matrix ($j=i+1$)
  In both cases, no multiplications are needed and $m(i,j)=0$.
- **IS:** Given $j-i \ge 2$ matrices and an optimal evalution tree T for them.
  - T's root must be a last product of left and right subterms $(M_i \cdots M_{k-1}) \cdot (M_k \cdots M_{j-1})$ for some $i < k < j$, with cost $r[i]r[k]c[j-1]$.
  - Moreover, left and right subtree T_ℓ and T_r of the root must be optimal evaluation trees for subproblems $[i..k)$ and $[k..j)$; (otherwise can improve T)

  $\rightsquigarrow$ By IH, the cost of T_ℓ and T_r are given by $m(i,k)$ and $m(k,j)$

  $\rightsquigarrow$ $m(i,j) = $ cost of T

Matrix-Chain Multiplication – Step 4: Memoization

- Write **recursive** function to compute recurrence
- But *memoize* all results!   (symbol table: subproblem $\mapsto$ optimal cost)

$\rightsquigarrow$ First action of function: check if subproblem known
  - If so, return cached optimal cost
  - Otherwise, compute optimal cost and remember it!

1. Subproblems
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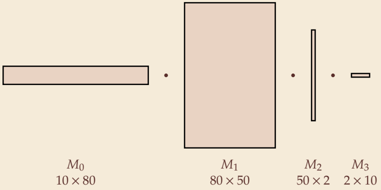
6. Backtrace

```
procedure totalMults(r[i..j), c[i..j)):
    if j − i ≤ 1
        return 0
    else
        best := +∞
        for k := i + 1, ..., j − 1
            m_l := cachedTotalMults(r[i..k), c[i..k))
            m_r := cachedTotalMults(r[k..j), c[k..j))
            m := m_l + m_r + r[i] · r[k] · c[j − 1]
            best := min{best, m}
        end for
        return best
```

$$m(i,j) = \begin{cases} 0 & \text{if } j-i \le 1 \\ \min\Big\{ m(i,k) + m(k,j) + r[i]\cdot r[k]\cdot c[j-1] \;:\; k \in [i+1..j) \Big\} & \text{otherwise} \end{cases}$$

```
procedure cachedTotalMults(r[i..j), c[i..j)):
    // m[0..n)[0..n) initialized to NULL at start
    if m[i][j] == NULL
        m[i][j] := totalMults(r[i..j), c[i..j))
    return m[i][j]
```

Matrix-Chain Multiplication – Example Memoization



$n = 4$

$r[0..n) = [10, 80, 50, 2]$

$c[0..n) = [80, 50, 2, 10]$

$m[i][j]$

$i \setminus j$	0	1	2	3	4
0	0	0	40000	9600	9800
1	—	0	0	8000	9600
2	—	—	0	0	1000
3	—	—	—	0	0
4	—	—	—	—	0

Matrix-Chain Multiplication – Runtime Analyses

```
procedure totalMults(r[i..j), c[i..j)):
    if j − i ≤ 1
        return 0
    else
        best := +∞
        for k := i + 1, ..., j − 1
            m_l := cachedTotalMults(r[i..k), c[i..k))
            m_r := cachedTotalMults(r[k..j), c[k..j))
            m := m_l + m_r + r[i] · r[k] · c[j − 1]
            best := min{best, m}
        end for
        return best
```

```
procedure cachedTotalMults(r[i..j), c[i..j)):
    // m[0..n)[0..n) initialized to NULL at start
    if m[i][j] == NULL
        m[i][j] := totalMults(r[i..j), c[i..j))
    return m[i][j]
```

- With memoization, compute each subproblem at most once
- nonrecursive cost (totalMults): $O(j - i) = O(n)$
- Number of subproblems $[i..j)$ for $0 \le i \le j \le n$

$$\sum_{0 \le i \le j \le n} 1 \;=\; \sum_{i=0}^{n} \sum_{j=i}^{n} 1 \;=\; \Theta(n^2)$$

$\rightsquigarrow$ total running time $O(n^3)$

Matrix-Chain Multiplication – Step 5: Table Filling

- Recurrence induces a DAG on subproblems (who calls whom)
  - Memoized recurrence traverses this DAG (DFS!)
  - We can slightly improve performance by systematically computing subproblems following a fixed topological order

1. Subproblems
2. Guess!
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- **Topological order** here: by **increasing length** $\ell = j - i$, then by i

```
procedure totalMultsBottomUp(r[0..n), c[0..n)):
    m[0..n)[0..n) := 0 // initialize to 0
    for ℓ = 2, 3, ..., n // iterate over subproblems ...
        for i = 0, 1 ..., n − ℓ // ... in topological order
            j := i + ℓ
            m[i][j] := +∞
            for k := i + 1, ..., j − 1
                q := m[i][k] + m[k][j] + r[i] · r[k] · c[j − 1]
                m[i][j] := min{m[i][j], q}
    return m[0..n)[0..n)
```

- Same Θ-class as memoized recursive function
- In practice usually substantially faster
  - lower overhead
  - predictable memory accesses

Matrix-Chain Multiplication – Step 6: Backtracing

- So far, only determine the **cost** of an optimal solution
  - But we also want the solution itself
- By *retracing* our steps, we can determine/construct one!
- Here: output a parenthesized term recursively

```
procedure matrixChainMult(r[0..n), c[0..n)):
    m[0..n)[0..n) := totalMultsBottomUp(r[0..n), c[0..n))
    return traceback([0..n))

procedure traceback([i..j)):
    if j − i == 1
        return M_i
    else
        for k := i + 1, . . . , j − 1
            q := m[i][k] + m[k][j] + r[i] · r[k] · c[j − 1]
            if m[i][j] == q
                return (traceback([i..k))) · (traceback([k..j)))
        end for
    end if
```

1. Subproblems
2. Guess!
3. DP Recurrence
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5. Table Filling
6. Backtrace

- follow recurrence a second time
- always have for running time: backtracing = O(computing M)

$\rightsquigarrow$ computing optimal cost and computing optimal solution have same complexity

- speedup possible by remembering correct guess k for each subproblem

Summary: The 6 Steps of Dynamic Programming

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

1. Define **subproblems** and how **original problem** is solved
2. What part of solution to **guess**?
3. Set up **DP recurrence** for quality/cost of solution
   - ⇝ Prove **correctness** here: induction over subproblems following recurrence
   - ⇝ Analyze running **time complexity** here: #subproblems · non-recursive time

— *(Basically) cookie-cutter approach from here on* —

4. Recursive implementation with **Memoization**: mutually recursive functions with cache

or

5. Bottom-up **table filling**: define topological order of subproblem dependency graph
6. **Backtracing** to reconstruct optimal solution: Recursively retrace cost recurrence

12.3 Greedy as Special Case of DP

Dynamic Greedy

- Every Greedy Algorithm can also be seen as a DP algorithm **without guessing**

$\rightsquigarrow$ For new problems, it can help to first follow the DP roadmap and then check if we can select the "correct" guess without local brute force

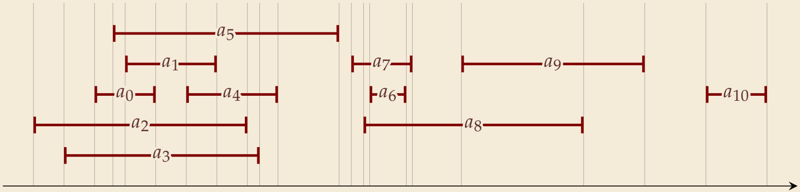
- If so, we then recurse on a single branch of subproblems

$\rightsquigarrow$ Greedy Algorithm doesn't need memoization or bottom-up table filling, but can do direct recursion instead

Recall Unit 11

The Activity selection problem

- **Activity Selection:** scheduling for *single* machine, jobs with *fixed* start and end times
  pick a *subset* of jobs without *conflicts*
  Formally:
  - **Given:** Activities $A = \{a_0, \ldots, a_{n-1}\}$, each with a start time s_i and finish time f_i $(0 \leq s_i < f_i < \infty)$
  - **Goal:** Subset $I \subseteq [0..n)$ of tasks such that $i, j \in I \wedge i \neq j \implies [s_i, f_i) \cap [s_j, f_j) = \emptyset$
    and $|I|$ is maximal among all such subsets
  - We further assume that jobs are sorted by finish time, i. e., $f_0 \leq f_1 \leq \cdots \leq f_{n-1}$
    (if not, easy to sort them in $O(n \log n)$ time)



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DP Algorithm for Activity Selection

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

1. **Subproblems:** $A_{i,j} = \{a_\ell \in A : s_\ell \geq f_i \wedge f_\ell \leq s_j\}$
   (after a_i finishes and before a_j begins)
   Original problem: $A_{-1,n}$ with dummy tasks $f_{-1} = -\infty$, $s_n = +\infty$
2. **Guess:** Task $k \in I^*$
3. **DP Recurrence:** Denote $c(i,j) = |I^*(A_{i,j})|$ = maximum #independent tasks in $A_{i,j}$

$$\rightsquigarrow \quad c(i,j) = \begin{cases} 0\,, & \text{if } A_{i,j} = \emptyset; \\ \max\{c(i,k) + c(k,j) + 1 : a_k \in A_{i,j}\} & \text{otherwise.} \end{cases}$$

4.–6. ***Omitted*** (could be done following the standard scheme)

- Problem-specific insight from Unit 11 $\rightsquigarrow$ Can always use $k = \min\{k : a_k \in A_{ij}\}$ (earliest finish time)

  No guess needed!

12.4 The Bellman-Ford Algorithm

Recall Shortest Paths

- **Single Source Shortest Path Problem (SSSPP)**
  - **Given:** directed, edge-weighted, simple graph $G = (V, E, c)$ with edge costs $c : E \to \mathbb{R}$, a start vertex $s \in V$
  - **Goal:** a data structure that reports for every $v \in V$:
    $\delta_G(s, v)$: the shortest-path distance from s to v
    $\texttt{spath}(v)$: a shortest path from s to v (if it exists)
- $\delta_G(s, v) \;=\; \boxed{\inf\big(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk in } G\}\big)}$
  - Write δ instead of δ_G when graph clear from context
- Here: Assume **negative-weight edges** are present (otherwise Dijkstra suffices)
  - but for now: assume there is **no negative cycle**

  $\rightsquigarrow$ $\delta(s, v) > -\infty$ and can restrict to shortest **paths** (not walks)

Shortest Paths as DP – Last Edge Decomposition

- Idea: Every nontrivial shortest path has a **last edge**. *We don't know which; so guess!*
  - $\rightsquigarrow$ Subproblems: for $w \in V$, compute $\delta(s, w)$.
  - $\rightsquigarrow$ Recurrence: $\delta(s, w) = \min\{\delta(s, v) + c(vw) : vw \in E\}$

subproblem dependency graph is isomorphic to G^T! $\rightsquigarrow$ doesn't work in general

$\rightsquigarrow$ Yields usable (terminating!) algorithm *iff* G is a DAG.

To break the cycles, let's turn them into a helix!

- Need to build "layers" in the subproblem dependency graph, so that edges can't go back up.
- **Subproblems:** (w, ℓ) for $w \in V$, $\ell \in [0..n)$, compute $\delta_{\le \ell}(s, w)$
  where $\delta_{\le \ell}(s, v) = \min\big(\{+\infty\} \cup \{c(w) : w \text{ an } s\text{-}v\text{-walk with } \le \ell \text{ edges}\}\big)$
- **Original problems:** $\ell = n - 1$ (without negative cycles, paths suffice)
- **Recurrence:** $$\delta_{\le \ell}(s, w) = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \neq w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\le \ell - 1}(s, v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$$

Shortest Paths as DP – Length Layers

Hold On – What about negative cycles?

- The recurrence for $\delta_{\leq \ell}$ seems to work fine with *negative* edges

  But G could contain a **negative-weight cycle** C . . .

$$\delta_{\leq \ell}(s,w) = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \neq w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\leq \ell-1}(s,v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$$

Isn't that a contradiction to the non-existence of shortest walks?

- No. If we restrict the length, shortest walks always exist.
- But: If there is a negative cycle $C[0..k]$ with paths $s \rightsquigarrow C$ and $C \rightsquigarrow w$,
  then $\delta_{\leq \ell}(s,w) > \delta_{\leq \ell+k}(s,w) > \delta_{\leq \ell+2k}(s,w) > \cdots$ (and $\delta(s,w) = -\infty$)

$\rightsquigarrow$ We can *detect* if any negative cycle is reachable from s by including more layers $\ell \geq n$ and check if some vertex still improves.

  - *How many further layers do we need / when is it safe to stop?*

Detecting negative cycles

We can detect reachable negative cycles by including just the *single* extra layer $\ell = n$!

Lemma: $\exists w : \delta_{\le n}(s,w) < \delta_{\le n-1}(s,w)$ *iff* negative cycle reachable from s

Proof:

"$\Rightarrow$"

- If some vertex w improves further, i. e., $\delta_{\le n}(s,w) < \delta_{\le n-1}(s,w)$ a walk $W[0..n]$ with $c(W) = \delta_{\le n}(s,w)$ was the **shortest** way to reach w

$\rightsquigarrow$ W is a non-simple walk, i. e., it contains a cycle

- Let $P[0..k]$ be the path resulting from W by shortcutting all cycles $\rightsquigarrow$ $k \le n-1$

$\rightsquigarrow$ $c(P) \ \ge\ \delta_{\le n-1}(s,w) \ >\ \delta_{\le n}(s,w) \ =\ c(W)$

$\rightsquigarrow$ $\exists$ negative cycle reachable from s

"$\Leftarrow$"

- Conversely, let negative cycle $C[0..k]$ be reachable from s

$\rightsquigarrow$ $c(C) = \sum_{i=0}^{k-1} c(C[i]C[i+1]) < 0$

- Assume towards a contradiction that $\forall w : \delta_{\le n}(s,w) = \delta_{\le n-1}(s,w)$

$\rightsquigarrow$ $\forall vw \in E : \delta_{\le n-1}(s,w) \le \delta_{\le n-1}(s,v) + c(vw)$ (no update in layer $\ell = n$)

- summing this inequality over $C[0..k]$ yields (abbreviating $\delta(w) := \delta_{\le n-1}(s,w)$)

$$\sum_{i=1}^{k} \delta(C[i]) \ \le\ \sum_{i=1}^{k}\Big(\delta(C[i-1]) + c(C[i]C[i+1])\Big) \ =\ \sum_{i=0}^{k-1} \delta(C[i]) + \underbrace{\sum_{i=1}^{k} c(C[i]C[i+1])}_{=c(C)<0}$$

$\rightsquigarrow$ $0 \le c(C) < 0$ ϟ ■

Shortest Paths as DP – Template Algorithm

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

- Strictly following the template works …
  - Subproblem order: by increasing $\ell \in [0..n]$ and $v \in V$
  - Bottom-up table filling:

```
procedure shortestPathsDP(G, s):
    // Base case ℓ = 0:
    δ[0..n][0..n) := +∞ // δ[ℓ][v] will store δ_{≤ℓ}(s,v)
    δ[0][s] := 0
    for ℓ := 1,...,n // layer
        for w := 0,...,n−1 // vertex
            for vw ∈ E
                δ[ℓ][w] := min{δ[ℓ][w], δ[ℓ−1][v] + c(vw)}
    return δ
```

$$\delta_{\le\ell}(s,w) = \begin{cases} \infty & \text{if } \ell = 0 \text{ and } s \ne w \\ 0 & \text{if } \ell = 0 \text{ and } s = w \\ \min\{\delta_{\le\ell-1}(s,v) + c(vw) : vw \in E\} & \text{otherwise} \end{cases}$$

- … but some improvements are possible!
  - Iterating over *incoming* edges is not convenient
    - $\leadsto$ order of updates within layer ℓ doesn't matter $\leadsto$ iterate forwards!
  - only use final distances in the end; we waste space by keeping 2D array around
    - $\leadsto$ can actually just do updates in place, using a single array δ
    - $\leadsto$ Don't strictly solve subproblems (ℓ, v) any more! (but final result correct)

The Bellman-Ford Algorithm

```
procedure bellmanFord(G, s):
    dist[0..n) := +∞;  pred[0..n) := null
    dist[s] := 0
    for ℓ := 1, ..., n − 1
        for v := 0, ..., n − 1
            for (w, c) ∈ G.adj[v]
                if dist[w] > dist[v] + c
                    dist[w] := dist[v] + c
                    pred[w] := v // remember for backtrace
    for v := 0, ..., n − 1
        for (w, c) ∈ G.adj[v]
            if dist[w] > dist[v] + c
                return HAS_NEGATIVE_CYCLE
    return (dist, pred)
```

- Final algorithm (including shortest path tree via *pred*)
- **Correctness:**
  - by induction over loop iterations, show:
    (a) $dist[w] \le \delta_{\le \ell}(s, w)$ and if finite,
    (b) $dist[w]$ is $c(P)$ for some s-w-path
  - negative cycle detection from Lemma
- **Space:** $\Theta(n)$
- **Running time:** $O(n(n + m))$

Extensions:

- Can be implemented in $O(nm)$ time by removing unreachable vertices from consideration
- Instead of only detecting a negative cycle, we can return one;
  we can also explicitly find all vertices with $\delta(s, w) = -\infty$ (needs another traversal).
- Can terminate with smaller ℓ if no distance changed $\rightsquigarrow$ faster for some graphs

12.5 Making Change in Pre-1971 UK

Recall Unit 11

Greed For Change

The Change-Making Problem (a.k.a. Coin-Exchange Problem)

- **Given:** a set of integer denominations of coins $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$, target value $n \in \mathbb{N}_{\geq 1}$

(we have sufficient supply of all coins . . .)

- **Goal:** "fewest coins to give change n", i.e., multiplicities $c_1, \ldots, c_k \in \mathbb{N}_{\geq 0}$ with $\sum_{i=1}^{k} c_i \cdot w_i = n$ minimizing $\sum_{i=1}^{k} c_i$

For Euro coins, denominations are 1¢, 2¢, 5¢, 10¢, 20¢, 50¢, 1€, and 2€.

formally:	1	2	5	10	20	50	100	and 200.
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8

$\rightsquigarrow$ Simple greedy algorithm: largest coins first

- optimal time ($O(k)$ if coins sorted)
- is $\sum c_i$ minimal?

```
procedure greedyChange(w[1..k], n):
    // Assumes 1 = w[1] < w[2] < ··· < w[k]
    for i := k, k − 1, . . . , 1:
        c[i] := ⌊n/w[i]⌋
        n := n − c[i] · w[i]
    // Now n == 0
    return c[1..k]
```

5

Pre-Decimal English Coins

We discussed that for some (unwise) choices of denominations, Greedy cannot give optimal change.
Welcome to Britain until 1971!

British Pre-Decimal Coins:

- $\frac{1}{2}$ penny,
- 1 penny,
- 3 pence,
- 6 pence,
- shilling = 12 pence,
- florin = 24 pence
- half-crown = 30 pence
- crown = 60 pence
- pound = 240 pence
- guinea = $21 \cdot 12 = 252$ pence
  (obsolete as coin since 1816)

$\rightsquigarrow$ Greedy would give 48 pence as 30p + 12p + 6p

- obviously, 2 florins are more efficient

$\rightsquigarrow$ How to solve exactly?

As the old saying goes . . .

Where Greedy fails, DP prevails.

(but mind details, and how it scales)

Making Change by DP

Idea: Every solution must pick a first coin. Which one? Unclear, so guess!

- **Subproblems:** Change for $m \in [0..n]$ (with coins $w_1, \ldots, w_k$)
  Original problem $m = n$
- **Guess:** first coin w_i to use
- **Recurrence** $C(m)$ = smallest #coins to give change m

$$C(m) \;=\; \begin{cases} 0 & \text{if } m = 0 \\ 1 + \min\big\{C(m - w_i) : i \in [1..k] \wedge w_i \le m\big\} & \text{otherwise} \end{cases}$$

- **Bottom-up implementation & Backtrace**

```
procedure dpChange(w[1..k], n):
    C[0..n] := +∞
    C[0] := 0
    for m := 1, ..., n
        for i := 1, ..., k
            if w[i] ≥ m
                q := 1 + C[m − w[i]]
                C[m] := min{C[m], q}
    return C[n]
```

```
procedure tracebackChange(w[1..k], n):
    C[0..n] := dpChange(w[1..k], n)
    c[1..k] := 0 // coin multiplicities
    m := n
    while m > 0
        for i := 1, ..., k
            if w[i] ≥ m ∧ C[m] == 1 + C[m − w[i]]
                c[i] := c[i] + 1;   m := m − w[i]
    return c[1..k]
```

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

Making Change by DP – Analysis

- **Input:** denominations of coins
  $w_1 < w_2 < \cdots < w_k$ with $w_1 = 1$,
  target value $n \in \mathbb{N}_{\geq 1}$

```
procedure dpChange(w[1..k], n):
    C[0..n] := +∞
    C[0] := 0
    for m := 1, ..., n
        for i := 1, ..., k
            if w[i] ≥ m
                q := 1 + C[m − w[i]]
                C[m] := min{C[m], q}
    return C[n]
```

- **Space:** $\Theta(n)$

#subproblems

time per subproblem

- **Running Time:** $O(n \cdot k)$

How good is this running time?

- A linear function in both input numbers seems decent, right? (If k and n small, certainly Yes.)
  - Running time is also certainly a *polynomial* in n and k
- But: In terms of *computational complexity*, dpChange is an **exponential-time algorithm**!
  - Reason: We give the input **number** n in **binary**, so n is exponential in its ***input size***.

⚠ Must distinguish: *value* of a number (in the input) vs. *size* of the (encoding of the) input

⇝ dpChange is a ***pseudo-polynomial time*** algorithm

- Actually, the general making-change problem is NP-complete (!)

Knapsack

Let's look at slightly more interesting problem: *Knapsack* („*Rucksack*").

a.k.a. the burglar's problem

The 0/1-Knapsack Problem

- **Given:** k items with weights $w_1 \ldots, w_k \in \mathbb{N}_{\geq 1}$ and values $v_1, \ldots, v_k \in \mathbb{R}_{\geq 0}$;
  a weight budget $W \in \mathbb{N}$
- **Goal:** Subset $I \subseteq [1..k]$ such that $\sum_{i \in I} w_i \leq W$ with maximum $\sum_{i \in I} v_i$.

Variant closer to Making change: Can use each item several times

- Recall from tutorials: Greedy fails miserably in general.

$\rightsquigarrow$ Let's try DP!

  - **Subproblems:** $B \in [0..W]$, best value with total weight $\leq B$
  - **Guess:** first item i with $w_i \leq B$.

  ϟ Subproblem not of same type since w_i no longer there!

  $\rightsquigarrow$ 2^k possible "states" to be in (items already used) (***0/1***-Knapsack)

  ϟϟ need a table of size $W \cdot 2^k$ ... *might as well do brute force then!*

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

Knapsack by DP

$\rightsquigarrow$ *Force order to consider items in!*

- Let's refine the guessing part to
  Guess: Whether or not to include the *last* item (k)
  - $\rightsquigarrow$ For subproblem, restrict to items $1, \ldots, k-1$ (in either case)

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

$\rightsquigarrow$ **Subproblems:** (ℓ, B) for $\ell \in [1..k]$ and $B \in [0..W]$

$V(\ell, B) = \max_I \sum_{i \in I} v_i$ over sets of items $I \subset [1..\boldsymbol{\ell}]$ with $\sum_{i \in I} w_i \le \boldsymbol{B}$

Original problem corresponds to $V(k, W)$

- **Recurrence:**

$$V(\ell, B) = \begin{cases} 0 & \text{if } \ell = 1 \wedge w_1 > B \\ v_1 & \text{if } \ell = 1 \wedge w_1 \le B \\ \max\Big\{ \overbrace{v_\ell + V(\ell-1, B-w_k)}^{\text{take item } \ell},\ \overbrace{V(\ell-1, B)}^{\text{don't take } \ell} \Big\} & \text{otherwise} \end{cases}$$

Cookie-Cutter Steps 4.–6. ***Omitted***

- $V(\ell, \cdot)$ only needs $V(\ell-1, \cdot)$ $\rightsquigarrow$ two arrays $V[0..W]$ and $V_{\text{prev}}[0..W]$ suffice

$\rightsquigarrow$ $\Theta(W)$ **space**, $\Theta(W \cdot k)$ **time** (pseudo-polynomial algorithm)

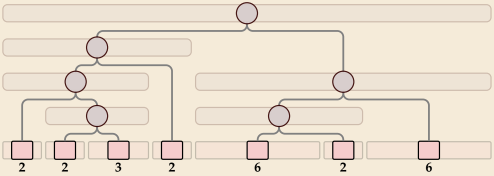
12.6 Optimal Merge Trees & Optimal BSTs

Recall Unit 4

Good merge orders

Let's take a step back and breathe.

- Conceptually, there are two tasks:
  1. Detect and use existing runs in the input $\rightsquigarrow$ $\ell_1, \ldots, \ell_r$ (easy) ✓
  2. **Determine a favorable *order of merges* of runs** ("automatic" in top-down mergesort)



Merge cost = total area of ▭
= total length of paths to all array entries
$$= \sum_{w \text{ leaf}} \mathit{weight}(w) \cdot \mathit{depth}(w)$$

$\rightsquigarrow$ *optimal* merge tree
= optimal *binary search tree*
for leaf weights $\ell_1, \ldots, \ell_r$
(optimal expected search cost)

well-understood problem
with known algorithms

Optimal Alphabetic Trees

"well-understood problem with known algorithms" ... let's make it so 😉

- **Given:** Leaf weights $\ell_0, \ldots, \ell_n$ normalized to $\ell_0 + \cdots + \ell_n = 1$
- **Goal:** Binary search tree T with $n + 1$ null pointers $L_0, \ldots, L_n$, such that
  $$c(T) \;:=\; \sum_{i=1}^{n} \ell_i \cdot \operatorname{depth}_T(L_i) \quad \text{is minimized}$$
- **Equivalent interpretations:**
  1. *Optimal Static BST* with keys $1, 2, \ldots, n$
     $\rightsquigarrow$ leaf L_i reached when searching for $i + 0.5$ $\rightsquigarrow$ $c(T)$ *expected* cost of *unsuccessful search*
     #comparisons
  2. *Alphabetic code* for $\sigma = n + 1$ symbols; like Huffman code, but *codewords must retain order* (if $i < j$ then the codeword for i lexicographically smaller than codeword for j)
     $\rightsquigarrow$ $c(T)$ expected codeword length
     - Inherit lower bound from Huffman codes: $c(T) \;\ge\; \mathcal{H}$ with $\mathcal{H} \;=\; \sum_{i=0}^{n} \ell_i \cdot \log_2\left(\frac{1}{\ell_i}\right)$
  3. *Merge tree* for adaptive sorting; $c(T)$ = merge cost per element.
     - Via Peeksort or Powersort know methods to achieve $c(T) \le \mathcal{H} + 2$
     - But neither are in general optimal

Optimal Alphabetic Trees by DP

- **Guess:** (Key in) root $r \in [1..n]$ of BST T (= #leaves in left subtree)
- **Subproblems:** $[i..j)$ for $0 \le i < j \le n+1$
  $C(i,j)$ = cost of opt. BST with leaf weights $\ell_i, \ldots, \ell_{j-1}$
  Original problem: $C(0, n+1)$
- **Recurrence:**

all leaves in subtree pay 1 at root. . .

$$C(i,j) \;=\; \begin{cases} 0 & \text{if } j-i=1 \\ \ell_i + \cdots + \ell_{j-1} + \min\big\{ C(i,r) + C(r,j) : r \in [i+1..j-1] \big\} & \text{otherwise} \end{cases}$$

. . . plus cost to continue in left/right subtree

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

$\rightsquigarrow$ Obtain a $O(n^3)$ time and $O(n^2)$ space algorithm

Optimal Binary Search Trees

- Algorithm can be generalized to Optimal BSTs when also internal nodes have weights
  - Same DP subproblems
- Running time can be reduced to $O(n^2)$ using *quadrangle inequality*
  - Intuitively: When adding more weight in right subtree, optimal root cannot move left.
  - Requires to remember r for each subproblem
- For original alphabetic tree problem, can actually find optimal tree in $O(n \log n)$ time with a much more intricate algorithm

12.7 Edit Distance

Edit Distance

Our last DP application here: (algorithmic foundation of) `diff`!

- `diff` is a classic Unix tool to compare two text files
- routinely used in version control systems such as `git`
- abstract problem: measure how different two strings are
  - We've seen *Hamming distance* ...
    But how to deal with strings of different lengths?
  - how to match common parts that are far apart?
  - `diff` works line-oriented, but we will formulate the problem character oriented

Edit Distance Problem

- **Given:** String $A[0..m)$ and $B[0..n)$ over alphabet $\Sigma = [0..\sigma)$.
- **Goal:** $d_{\text{edit}}(A, B) =$ minimal # symbol operations to transform A into B
  operations can be insertion/deletion/substitution of single character

Edit Distance Example

Example: edit distance $d_{\text{edit}}(\texttt{algorithm}, \texttt{logarithm})$?

```
algorithm

logarithm
```

```
0123456789
al·gorithm
-|+|X|||||
·logarithm
```

Edit Distance by DP

1. **Subproblems:** (i,j) for $0 \le i \le m$, $0 \le j \le n$ compute $d_{\text{edit}}(A[0..i), B[0..j))$
2. **Guess:** What to do with last positions? (insert/delete/(mis)match)
3. **Recurrence:** $D(i,j) = d_{\text{edit}}(A[0..i), B[0..j))$

$$D(i,j) \;=\; \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \begin{cases} D(i-1,j)+1, \\ D(i,j-1)+1, \\ D(i-1,j-1)+\big[A[i-1] \neq B[j-1]\big] \end{cases} & \text{otherwise} \end{cases}$$

⇝ $O(nm)$ space and time
space can be improved to $O(\min\{n,m\})$ by remembering only 2 rows or columns

- An optimal *edit script* can be constructed by a backtrace

Generalized Edit Distances

- The variant we discussed is also called *Levenshtein distance*
  - all operation have cost 1
- we can directly give each of the following its **own cost** in our DP algorithm
  - deleting an occurrence of $a \in \Sigma$
  - inserting an $a \in \Sigma$
  - substituting $a \in \Sigma$ for $b \in \Sigma$
- Extensions of the algorithm can support:
  - **free** insert/delete at beginning/end of a string
  - *affine gap costs*, i. e., inserting/deleting k **consecutive** chars costs $c \cdot k + d$ for constants c and d
- extensions widely used to find approximate matches, e. g., in DNA sequences

  $\rightsquigarrow$ *Algorithms of Bioinformatics*

Dynamic Programming – Summary

1. Subproblems
2. Guess!
3. DP Recurrence
4. Memoization
5. Table Filling
6. Backtrace

- Versatile and powerful algorithm design paradigm
- Once key idea (recurrence) clear, implementation rather straight-forward